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A Further Note on Construction of Graeco Latin Square of Order 2n + 1

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SUMMARY

A generalized method of constructing graeco latin square of any odd order (>– 3) is given. The procedure is also illustrated for constructing all the distinct graeco latin squares of order 3.

Key-words: Orthogonal latin squares, Initial block, Construction

1. Introduction

Das and Dey [1] have presented a method of constructing a pair of orthogonal latin squares of any odd order by developing two initial rows. By superimposing those orthogonal latin squares one may get the graeco latin square. In the present note, a more generalized method of constructing graeco latin squares of any odd order has been presented which enables one to obtain several graeco latin squares from the given latin square. The results are presented in Section 2. Further the distinct graeco latin squares of order 3, obtained by applying the proposed method are also presented.

2. Main Results

Let m (= 2n + 1) be the number of symbols coded by (1, 2, ..., m) and is denoted by the set S = {1, 2, ..., m}. The latin squares are obtained by generating the initial blocks mod m.

Theorem 2.1 : A graeco latin square (equivalently a pair of orthogonal latin squares) of order m can be obtained by developing the following two initial rows.

$$R_1 : i_1\ i_2\ i_3 \ldots i_m \bmod m$$

$$R_2 : j_1\ j_2\ j_3 \ldots j_m \bmod m \qquad (2.1)$$

provided (i) $i_k \neq i_1$ if $k \neq 1$

(ii) $j_k \neq j_1$ if $k \neq 1$

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(iii) $i_k + j_k = p, \ k = 1, 2, \ldots, m$ (2.2)

where i_k, j_k and p are the members of the set S.

Proof: By developing, the two initial rows given above one may obtain two square arrays of order m and are denoted by A and B respectively. The (k, l)th position of A and B are respectively filled by

$$a_{kl} = i_l + (k-1) \bmod m \text{ and}$$

$$b_{kl} = p - i_l + (k-1) \bmod m \tag{2.3}$$

To show that A and B are latin squares and are mutually orthogonal consider the following :

Let $k \neq l \neq r \neq s$, then we have

$$a_{kl} = i_l + (k-1) \bmod m$$

$$a_{kr} = i_r + (k-1) \bmod m$$

$$a_{sl} = i_l + (s-1) \bmod m \tag{2.4}$$

From (2.4), we can clearly show that no two elements in the same row and in the same column are equal and hence A is a latin square of order m. Similarly we can show that B is also a latin square of order m.

On superimposing A and B one may get a square array of order m and is denoted by G. The (k, l)th position of G is obtained as

$$g_{kl} = (a_{kl}, b_{kl}) \tag{2.5}$$

where a_{kl} and b_{kl} as defined in (2.3).

The elements a_{kl} and b_{kl} are equal if

$$2i_l - p = 0 \bmod m \tag{2.6}$$

But this is true only when $i_l = p/2$. Hence the column in which $i_l = p/2$, the elements a_{kl} and b_{kl} are equal and not in other columns.

Further we have to show that no two positions of G have been filled by the same elements.

From the definition of A and B no two elements in the same row and in the same column are equal. Therefore it is enough to prove that the elements in different rows and in different columns are not equal.

Let $\quad k \neq r \quad$ and $\quad l \neq s$ (2.7)

By the definition of G we have,

$g_{kl} = (a_{kl}, b_{kl}) = (i_l + (k - 1), p - i_l + (k - 1)) \bmod m$ and

$g_{rs} = (a_{rs}, b_{rs}) = (i_s + (r - 1), p - i_s + (r - 1)) \bmod m$

Let $g_{kl} = g_{rs}$ It is true only if

$$i_l - i_s = r - k \text{ and } i_l - i_s = k - r \tag{2.8}$$

From (2.8) we have

$$k = r \tag{2.9}$$

From (2.8) and (2.9) we have shown that $i_l - i_s = 0$. That is $i_l = i_s$, which is true only when

$$l = s \tag{2.10}$$

From (2.9) and (2.10) we have obtained k = r and l = s, which is a contradiction to the assumption (2.7). Hence, the proof.

Remark 2.1: The theorem of Das and Dey [1] is a particular case of the Theorem 2.1 given above, in which $i_l = l - 1$, $j_l = m + 1 - l$ and $p = 0$.

Remark 2.2: For $k \neq r$ and $l \neq s$, one can easily show from equation (2.8) that the two elements g_{rs} and g_{kl} of a graeco latin square are equal, provided $r - k = k - r$. That is $2(k - r) \bmod m = 0$, which is true only when m is an even number. Hence the proposed method is valid only for the construction of graeco latin squares of odd order and is not valid for the construction of graeco latin squares of even orders.

Theorem 2.2: From a given latin square, m distinct graeco latin squares can always be obtained.

Proof: Let i_k be the kth element of the initial block of the given latin square. From Theorem 2.1 one can easily obtain the graeco latin square with kth element of the initial block as (i_k, j_k) such that $i_k + j_k \bmod m = p$, where p takes any value in the set $S = \{1, 2, ..., m\}$. Hence the proof.

Theorem 2.3: There always exists $((m + 1)! - m!)$ distinct graeco latin squares of order m.

Proof: Let $(i_1\ i_2 \ldots i_m)$ be the initial row of a latin square of order m. By permutating the initial row one can obtain m! distinct initial rows which leads to m! distinct latin squares of order m. By applying the Theorem 2.2 we can always construct $m! \times m$ $(= (m + 1)! - m!)$ graeco latin squares.

Example

Let m (=3) be the number of symbols, denoted by S = (1, 2, 3). It is given that m = 3, we can construct m! = 3! =6 latin squares following Theorem 2.2 and (m + 1)! – m! = 4! – 3! = 18 graeco latin squares following Theorem 2.3.

The latin squares are denoted by Li, i = 1, 2, ..., 6 and are given as

L1 = (1 2 3)
L2 = (1 3 2)
L3 = (2 1 3)
L4 = (2 3 1)
L5 = (3 1 2)
L6 = (3 2 1)

The graeco latin squares are denoted by Gi, i = 1, 2, ..., 18 and are given as

G1 = [(1, 3) (2, 2) (3, 1)]	G10 = [(2, 2) (3, 1) (1, 3)]
G2 = [(1, 1) (2, 3) (3, 2)]	G11 = [(2, 3) (3, 2) (1, 1)]
G3 = [(1, 2) (2, 1) (3, 3)]	G12 = [(2, 1) (3, 3) (1, 2)]
G4 = [(1, 3) (3, 1) (2, 2)]	G13 = [(3, 1) (1, 3) (2, 2)]
G5 = [(1, 1) (3, 2) (2, 3)]	G14 = [(3, 2) (1, 1) (2, 3)]
G6 = [(1, 2) (3, 3) (2, 1)]	G15 = [(3, 3) (1, 2) (2, 1)]
G7 = [(2, 2) (1, 3) (3, 1)]	G16 = [(3, 1) (2, 2) (1, 3)]
G8 = [(2, 3) (1, 1) (3, 2)]	G17 = [(3, 2) (2, 3) (1, 1)]
G9 = [(2, 1) (1, 2) (3, 3)]	G18 = [(3, 3) (2, 1) (1, 2)]

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